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MEAN SQUARE INEQUALITIES FOR CHORDS OF CONVEX SETS

BY

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ABSTRACT

This paper is concerned with establishing lower bounds for the integrals of the square of the lengths of area and perimeter bisecting chords of planar convex sets. The results obtained provide verification of two recent conjectures of Lutwak. When combined with the known upper bounds for these integrals they yield the classical isoperimetric inequality. The main proof technique involves estimation of the winding numbers of the locus of the midpoints of the chords concerned.

1. Introduction

The initial motivation for this paper arises from some results of Lutwak [8]. He established some integral inequalities involving the lengths of perimeter (and area) bisecting chords of planar convex sets. If K is such a set and $\theta \in [0, 2\pi]$ we denote by $L(K;\theta)$ (and $A(K;\theta)$) the lengths of the perimeter (and area) bisectors of K which make angle θ with the horizontal. Two of the inequalities established by Lutwak are

(1) $$\left\{\frac{1}{2\pi}\int_0^{2\pi} L^2(K;\theta)d\theta\right\}^{\frac{1}{2}} \leqq L(K)/\pi$$

and

(2) $$\left\{\frac{1}{2\pi}\int_0^{2\pi} A^2(K;\theta)d\theta\right\}^{\frac{1}{2}} \leqq L(K)/\pi;$$

here $L(K)$ is the length of the perimeter of K. He showed that in both cases equality holds if and only if K is a disc. More recently it was shown in [3] that Lutwak's inequalities can be extended to fairly arbitrary families of chords. In fact if $\mathcal{F}$ is any family of chords of K such that for each direction θ there is

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precisely one member $f(\theta) \in \mathcal{F}$ lying in that direction and if the chords $f(\theta)$ vary continuously then

$$\left\{\frac{1}{2\pi}\int_0^{2\pi}\sigma^2(\theta)d\theta\right\}^{\frac{1}{2}} \leqq L(K)/\pi \tag{3}$$

where $\sigma(\theta)$ is the length of $f(\theta)$. Here equality occurs if and only if K has constant width and $f(\theta)$ is a chord of maximal length in direction θ.

Our present aim is to establish lower bounds for the integrals in (1) and (2). We shall prove the following two results.

THEOREM 1. *Let K be a plane convex set with area $A(K)$, then*

$$\int_0^{2\pi} L^2(K;\theta)d\theta \geqq 8A(K)$$

with equality if and only if K is centrally symmetric.

THEOREM 2. *Let K be a plane convex set with area $A(K)$, then*

$$\int_0^{2\pi} A^2(K;\theta)d\theta \geqq 8A(K)$$

with equality if and only if K is centrally symmetric.

We notice that combination of these results with (1) and (2) yields the classical isoperimetric inequality

$$4\pi A(K) \leqq L^2(K).$$

Also it follows that

$$\max\{L(K;\theta): \theta \in [0,2\pi]\} \geqq 2\sqrt{A(K)/\pi} \tag{4}$$

and

$$\max\{A(K;\theta): \theta \in [0,2\pi]\} \geqq \sqrt{A(K)/\pi} \tag{5}$$

with equality in either case occurring if and only if K is a disc. To see the equality conditions notice that for equality we certainly need K centrally symmetric. But then the perimeter and area bisecting chords are all bisected by the centre of K. Of course, we also require $L(K;\theta)$ (or $A(K;\theta)$) to be constant in order to obtain equality and so K must be a disc. The inequalities (4) and (5) were conjectured by Lutwak at the 1980 Oberwolfach "Konvexe Körper" conference. They are in fact analogues of the Herda inequalities (see [1, 2, 4, 6, 7, 11]).

2. Preliminary results and observations

The proofs of Theorems 1 and 2 will arise from a study of the winding numbers of certain closed curves contained within the convex set K. A key result which we shall use occurs in [3, equation (20)] and we shall state it here as a lemma. A family $\mathcal{F}$ of chords with the properties described in the introduction will be called a *simple* family. A *smooth* curve is one which is continuously differentiable.

LEMMA 1. *Let $\mathcal{F}$ be a simple family of chords of K and let $f(\theta)\in\mathcal{F}$ be the chord in direction θ having length $\sigma(\theta)$*. Let *$M(\mathcal{F})$ be the locus of mid-points of the chords of $\mathcal{F}$ and assume that $M(\mathcal{F})$ is piecewise smooth. Then*

$$A(K)-\int_{z\notin M(\mathcal{F})} w(z,M(\mathcal{F}))dz=\frac{1}{8}\int_0^{2\pi}\sigma^2(\theta)d\theta. \tag{6}$$

Here $w(z,M(\mathcal{F}))$ is the winding number of the closed curve $M(\mathcal{F})$ about the point z. We shall always assume that the orientation of $M(\mathcal{F})$ corresponds to the chords of $\mathcal{F}$ rotating in an anti-clockwise direction. In the case when $M(\mathcal{F})$ is a simple closed curve the integral on the left side of (6) is plus or minus the area enclosed by $M(\mathcal{F})$, depending on whether the orientation of $M(\mathcal{F})$ is respectively anti-clockwise or clockwise.

We shall denote by $\mathcal{G}$ (and $\mathcal{H}$) the family of perimeter (and area) bisecting chords of K. If $\mathcal{F}=\mathcal{G}$ we shall put $M(K)=M(\mathcal{F})$ and if $\mathcal{F}=\mathcal{H}$ put $N(K)=M(\mathcal{F})$. It is now clear from (6) that we may obtain the required inequalities by showing that all winding numbers of $M(K)$ and $N(K)$ are non-positive. In fact we shall do this in the case K is a polygon and then use an approximation argument.

LEMMA 2. *If $\mathcal{F}=\mathcal{G}$ or $\mathcal{H}$ then $M(\mathcal{F})$ is rectifiable.*

PROOF. First we introduce some notation which will be useful here and in the sequel. Let $\boldsymbol{m}(K;\theta)$ denote the mid-point of the chord of $\mathcal{F}$ in direction θ and let $\boldsymbol{x}(K;\theta)$ and $\boldsymbol{y}(K;\theta)$ be the end-points of this chord. We assume that $\boldsymbol{x}(K;\theta)$ and $\boldsymbol{y}(K;\theta)$ are chosen in such a way that they vary continuously as functions of θ. So we have

$$M(\mathcal{F})=\{\boldsymbol{m}(K:\theta):\theta\in[0,2\pi]\}.$$

Now choose $\theta_0<\theta_1<\cdots<\theta_n$ with $\theta_0=0$ and $\theta_n=2\pi$. Then for $i=0,1,\cdots,n-1$ we have

$$\|\boldsymbol{m}(K,\theta_{i+1})-\boldsymbol{m}(K,\theta_i)\|\leq\frac{1}{2}\|\boldsymbol{x}(K,\theta_{i+1})-\boldsymbol{x}(K,\theta_i)\|+\frac{1}{2}\|\boldsymbol{y}(K,\theta_{i+1})-\boldsymbol{y}(K,\theta_i)\|.$$

Consequently

$$\sum_{i=0}^{n-1} \|m(K,\theta_{i+1}) - m(K,\theta_i)\| \leqq \frac{1}{2}\sum_{i=0}^{n-1} \|x(K,\theta_{i+1}) - x(K,\theta_i)\|$$

$$+\frac{1}{2}\sum_{i=0}^{n-1} \|y(K,\theta_{i+1}) - y(K,\theta_i)\|$$

$$\leqq \frac{1}{2}L(K) + \frac{1}{2}L(K) = L(K),$$

and so $M(\mathscr{F})$ is rectifiable as required.

The next lemma enables us to use an approximation argument and thus allows us to concentrate on polygons for which, of course, the curves $M(\mathscr{F})$ are more manageable. In particular they are piecewise smooth.

LEMMA 3. *Let $(K_n)_{n=1}^{\infty}$ be a sequence of convex sets with $K_n \to K$ as $n \to \infty$ (in the Hausdorff metric). Then*

$$\int_0^{2\pi} L^2(K_n;\theta)d\theta \to \int_0^{2\pi} L^2(K;\theta)d\theta \qquad \text{as } n \to \infty \tag{7}$$

and

$$\int_0^{2\pi} A^2(K_n;\theta)d\theta \to \int_0^{2\pi} A^2(K;\theta)d\theta \qquad \text{as } n \to \infty. \tag{8}$$

PROOF. Only the first result will be established since the second can be proved in an analogous fashion.

We shall denote by $[x, y]$ the closed line segment joining the points x and y. For each n choose convex sets $C(K_n;\theta)$ and $D(K_n;\theta)$ such that

$$C(K_n;\theta) \cup D(K_n;\theta) = K_n$$

and $C(K_n;\theta) \cap D(K_n;\theta) = [x(K_n;\theta), y(K_n;\theta)]$ for all $\theta \in [0, 2\pi]$. We are working with the case $\mathscr{F} = \mathscr{G}$ and so

$$L(C(K_n;\theta)) = L(D(K_n;\theta))$$

for all n and θ. Now for fixed θ the chords $[x(K_n;\theta), y(K_n;\theta)]$ converge to a chord $l(\theta)$ of K in direction θ. Also the sets $C(K_n;\theta)$ and $D(K_n;\theta)$ converge to convex sets $C(K;\theta)$ and $D(K;\theta)$ respectively with

$$C(K;\theta) \cup D(K;\theta) = K$$

and

$$C(K;\theta) \cap D(K;\theta) = l(\theta).$$

But clearly $L(C(K;\theta)) = L(D(K;\theta))$ and so

$$l(\theta) = [\boldsymbol{x}(K;\theta), \boldsymbol{y}(K;\theta)].$$

Also there is some constant α, say, such that

$$L(K_n;\theta) \leqq \alpha$$

for all n and θ. So the result follows from an application of the dominated convergence theorem.

Next we note that because of the convexity of K, the one-sided tangents to K at the points $\boldsymbol{x}(K;\theta)$ and $\boldsymbol{y}(K;\theta)$ always exist. In the case $\mathscr{F} = \mathscr{G}$ we see that if the right (or left) hand unit tangent vectors $\boldsymbol{t}_1$ and $\boldsymbol{t}_2$ at $\boldsymbol{x}(K;\theta)$ and $\boldsymbol{y}(K;\theta)$ respectively are not parallel then $M(K)$ has a right (or left) hand tangent at $\boldsymbol{m}(K;\theta)$ and this tangent bisects $\boldsymbol{t}_1$ and $\boldsymbol{t}_2$. In fact in the case $K = P$ a polygon, the same observation shows that $M(P)$ is a closed polygonal curve.

If we still work with a polygon P but turn our attention to the case $\mathscr{F} = \mathscr{H}$ we find that $N(P)$ is a closed curve consisting of a finite number of hyperbolic arcs. Closer examination of this situation shows that the chords $[\boldsymbol{x}(P;\theta), \boldsymbol{y}(P;\theta)]$ are in fact tangent to their corresponding hyperbolas. Now let P be a polygon for which no two sides are parallel. Then if A_1 and A_2 are consecutive arcs of $N(P)$ with

$$A_1 \cap A_2 = \boldsymbol{m}(P;\theta)$$

then either A_1 and A_2 lie on the same side of $[\boldsymbol{x}(K;\theta), \boldsymbol{y}(K;\theta)]$ and $N(P)$ has a tangent at $\boldsymbol{m}(P;\theta)$ or the one-sided tangents to $N(P)$ at $\boldsymbol{m}(P;\theta)$ are in opposite directions and A_1 and A_2 are on different sides of $[\boldsymbol{x}(K;\theta), \boldsymbol{y}(K;\theta)]$. If we now consider an arbitrary set K and approximate it by means of a sequence of polygons it can be seen that the one-sided tangents to $N(K)$ at $\boldsymbol{m}(K;\theta)$ lie along the chords $[\boldsymbol{x}(K;\theta), \boldsymbol{y}(K;\theta)]$.

Our next aim is to investigate the convergence properties of certain winding numbers. To do this we recall from [9] that a polygonal path

$$\sigma = [\boldsymbol{x}_0, \boldsymbol{x}_1] \cup \cdots \cup [\boldsymbol{x}_{k-1}, \boldsymbol{x}_k]$$

is an ε-approximation to the rectifiable path s if there are points $\boldsymbol{y}_0, \cdots, \boldsymbol{y}_k$ in order along s such that for each r all points of the arc of s joining $\boldsymbol{y}_r$ and $\boldsymbol{y}_{r+1}$ are within distance ε of $\boldsymbol{x}_r$. The result we shall need is essentially contained in theorem 8.2 of [9, chapter VII]. It guarantees that if $\boldsymbol{z} \notin s$ and $\varepsilon > 0$ is sufficiently small then

$$w(z, s) = w(z, \sigma)$$

for any ε-approximation σ to s.

LEMMA 4. *Let $(P_n)_{n=1}^{\infty}$ be a sequence of polygons with $P_n \to K$ as $n \to \infty$. Then if $z \notin M(K)$ we have*

$$\lim_{n\to\infty} w(z, M(P_n)) = w(z, M(K)). \tag{9}$$

PROOF. We have seen in the proof of Lemma 3 that

$$\lim_{n\to\infty} \boldsymbol{x}(P_n;\theta) = \boldsymbol{x}(K;\theta)$$

and

$$\lim_{n\to\infty} \boldsymbol{y}(P_n;\theta) = \boldsymbol{y}(K;\theta).$$

In fact both of these are approached uniformly. To see this, assume for example that the first limit was only a pointwise limit. Then there is an $\varepsilon > 0$, a sequence $(n(i))_{i=1}^{\infty}$ with $n(i) \to \infty$ and a sequence $(\theta_{n(i)})_{i=1}^{\infty}$ such that

$$\|\boldsymbol{x}(P_{n(i)};\theta_{n(i)}) - \boldsymbol{x}(K;\theta_{n(i)})\| \geqq \varepsilon \tag{10}$$

for all i. We may assume, by taking subsequences if necessary, that $\theta_{n(i)} \to \theta$ as $i \to \infty$. But then

$$\boldsymbol{x}(P_{n(i)};\theta_{n(i)}) \to \boldsymbol{x}(K;\theta) \qquad \text{as } i \to \infty$$

and

$$\boldsymbol{x}(K;\theta_{n(i)}) \to \boldsymbol{x}(K;\theta) \qquad \text{as } i \to \infty.$$

This contradicts (10) and shows that the convergence is uniform. Consequently, given $\varepsilon > 0$ we can choose N such that

$$\|\boldsymbol{m}(P_n;\theta) - \boldsymbol{m}(K;\theta)\| < \varepsilon$$

for all $\theta \in [0, 2\pi]$ whenever $n \geqq N$.

We fix some $n \geqq N$ and let $\boldsymbol{v}_1, \cdots, \boldsymbol{v}_I$ be the vertices of the polygonal arc $M(P_n)$. Next choose points $\boldsymbol{x}_1, \cdots, \boldsymbol{x}_M$ in order along $M(P_n)$ so that

$$\{\boldsymbol{v}_i\}_{i=1}^{I} \subset \{\boldsymbol{x}_j\}_{j=1}^{M}$$

and

$$\|\boldsymbol{x}_j - \boldsymbol{x}_{j+1}\| < \varepsilon$$

for $j = 1, \cdots, M$ where $\boldsymbol{x}_{M+1} = \boldsymbol{x}_1$. For each integer j with $1 \leqq j \leqq M$ we may choose $\theta_j \in [0, 2\pi]$ such that

$$\boldsymbol{x}_j = \boldsymbol{m}(P_n;\theta_j);$$

we make this choice in such a way that $\theta_1 < \theta_2 < \cdots < \theta_M$. Now let $y = m(K;\theta)$ for some $\theta \in [\theta_i, \theta_{i+1}]$ then

$$\| m(P_n;\theta) - y \| < \varepsilon.$$

But $m(P_n;\theta) \in [x_i, x_{i+1}]$ and so

$$\| m(P_n;\theta) - x_i \| < \varepsilon.$$

Now we note that

$$M(P_n) = [x_1, x_2] \cup \cdots \cup [x_M, x_1]$$

and thus $M(P_n)$ is a 2ε-approximation to $M(K)$. The result now follows from the observations preceding the statement of the lemma.

It is clear that we could extend the notion of ε-approximation to paths consisting of a finite number of hyperbolic arcs. We would then obtain the analogue of Lemma 4 in the case $\mathcal{F} = \mathcal{H}$. That is, if $(P_n)_{n=1}^{\infty}$ is a sequence of polygons approaching K and $z \notin N(K)$ then

$$\lim_{n\to\infty} w(z, N(P_n)) = w(z, N(K)). \tag{11}$$

We notice that the curves $M(P)$ and $N(P)$ are traversed once as θ varies from 0 to π. It is quite possible that they are not simple closed paths, so we now investigate to what extent they can fail to be simple. First we show that no two distinct segments of $M(P)$ can contain a common subsegment. If this were possible we would have a segment $[a, b] \subset M(P)$ and angles $\theta_1, \theta_2, \phi_1, \phi_2 \in [0, \pi]$ with

$$a = m(P;\theta_1) = m(P;\theta_2) \qquad (\theta_1 < \theta_2)$$

and

$$b = m(P;\phi_1) = m(P;\phi_2) \qquad (\phi_1 < \phi_2).$$

Then we could find one-sided unit tangent vectors t_1, t_2, t_3 and t_4 at $x(P;\theta_1)$, $y(P;\theta_1)$, $x(P;\theta_2)$ and $y(P;\theta_2)$ respectively such that the angles between t_1 and t_2 and between t_3 and t_4 are both bisected by the line L through the origin parallel to the segment $[a, b]$. But the pair t_1, t_2 must be interlaced with the pair t_3, t_4 and so they cannot have a common bisector. This contradiction shows that two distinct segments of $M(P)$ intersect in at most one point.

The case $\mathcal{F} = \mathcal{H}$ is much easier to deal with. The tangential properties of the area bisecting chords clearly show that no two hyperbolic arcs of $N(P)$ can contain a common subarc. Consequently, any two distinct hyperbolic arcs of $N(P)$ intersect in at most four points.

So we have seen that the curves $M(P)$ and $N(P)$ each have at most a finite number of self-intersections. We now introduce a construction which reduces this number of self-intersections and thus, in the case of polygons, enables us to consider just the case where $M(P)$ (or $N(P)$) is a simple closed curve.

We assume $\mathcal{F} = \mathcal{G}$ or $\mathcal{H}$ and that K is a convex set for which there are angles θ_1, θ_2 with $\theta_1 < \theta_2$ and $\boldsymbol{m}(K;\theta_1) = \boldsymbol{m}(K;\theta_2)$. Let H_1 and H_2 be the closed half-planes containing $\boldsymbol{m}(K;\theta_1)$ which are determined by the lines containing $[\boldsymbol{x}(K;\theta_1), \boldsymbol{x}(K;\theta_2)]$ and $[\boldsymbol{y}(K;\theta_1), \boldsymbol{y}(K;\theta_2)]$ respectively. Then put

$$K(\theta_1, \theta_2) = K \cap H_1 \cap H_2$$

and note that the boundary of $K(\theta_1, \theta_2)$ contains the segments $[\boldsymbol{x}(K;\theta_1), \boldsymbol{x}(K;\theta_2)]$ and $[\boldsymbol{y}(K;\theta_1), \boldsymbol{y}(K;\theta_2)]$. Also if $Q = K(\theta_1, \theta_2)$ and $\theta_1 \leqq \theta \leqq \theta_2$ we have

$$\boldsymbol{m}(Q;\theta) = \boldsymbol{m}(K;\theta_1) = \boldsymbol{m}(K;\theta_2)$$

whereas if $\theta \in [0, \theta_1] \cup [\theta_2, \pi]$ we have

$$\boldsymbol{m}(Q;\theta) = \boldsymbol{m}(K;\theta).$$

Consequently $M(Q) \subset M(K)$ or $N(Q) \subset N(K)$ where in both cases these sets are thought of as oriented curves. An analogous construction produces the set $K[\theta_1, \theta_2]$ whose boundary contains the segments $[\boldsymbol{y}(K;\theta_2), \boldsymbol{x}(K;\theta_1)]$ and $[\boldsymbol{x}(K;\theta_2), \boldsymbol{y}(K;\theta_1)]$. In this case, if we put $R = K[\theta_1, \theta_2]$ we see that for $\theta_1 \leqq \theta \leqq \theta_2$ we have

$$\boldsymbol{m}(R;\theta) = \boldsymbol{m}(K;\theta)$$

whereas if $\theta \in [0, \theta_1] \cup [\theta_2, \pi]$ we have

$$\boldsymbol{m}(R;\theta) = \boldsymbol{m}(K;\theta_1) = \boldsymbol{m}(K;\theta_2).$$

Once again we have $M(R) \subset M(K)$ or $N(R) \subset N(K)$. Combining these two results we see that

$$M(K) = M(Q) \cup M(R)$$

or

$$N(K) = N(Q) \cup N(R).$$

In the case $K = P$ a polygon we see that repeated use of these constructions will yield polygons $Q_1, Q_2, \cdots, Q_t$ such that either

$$M(P) = \bigcup_{i=1} M(Q_i)$$

or

$$N(P)=\bigcup_{i=1} N(Q_i)$$

and such that for each i, $M(Q_i)$ or $N(Q_i)$ is a simple closed curve. Obviously, the above alternatives depend on whether $\mathcal{F}=\mathcal{G}$ or $\mathcal{F}=\mathcal{H}$. We thus conclude that if $\mathcal{F}=\mathcal{G}$ and $z\notin M(P)$ we have

$$w(z;M(P))=\sum_{i=1}^{t} w(z;M(Q_i)) \tag{12}$$

whereas if $\mathcal{F}=\mathcal{H}$ and $z\notin N(P)$ we obtain

$$w(z;N(P))=\sum_{i=1}^{t} w(z;N(Q_i)). \tag{13}$$

The next lemma which we shall use to characterize the curves for which equality holds, shows that in order to prove the two theorems it suffices to focus our attention on sets of the form $K(\theta_1,\theta_2)$ and $K[\theta_1,\theta_2]$.

Lemma 5. *Let $\mathcal{F}=\mathcal{G}$ or $\mathcal{H}$ and assume that there are angles θ_1, θ_2 with $0\leqq\theta_1<\theta_2\leqq\pi$ such that $\boldsymbol{m}(K;\theta_1)=\boldsymbol{m}(K;\theta_2)$. We put $Q=K(\theta_1,\theta_2)$ and $R=K[\theta_1,\theta_2]$. Then if $\mathcal{F}=\mathcal{G}$ and*

$$\int_0^{2\pi} L^2(C;\theta)d\theta\geqq 8A(C)$$

for $C=Q$ and R, we have

$$\int_0^{2\pi} L^2(C;\theta)d\theta-8A(C)\leqq\int_0^{2\pi} L^2(K;\theta)d\theta-8A(K) \tag{14}$$

for $C=Q$ and R. Similarly if $\mathcal{F}=\mathcal{H}$ and

$$\int_0^{2\pi} A^2(C;\theta)d\theta\geqq 8A(C)$$

for $C=Q$ and R, we have

$$\int_0^{2\pi} A^2(C;\theta)d\theta-8A(C)\leqq\int_0^{2\pi} A^2(K;\theta)d\theta-8A(K) \tag{15}$$

for $C=Q$ and R.

Proof. We shall just establish (14) in the case $C=Q$; the remaining three possibilities can be dealt with in the same fashion.

Let A denote the area of the triangle with vertices $\boldsymbol{x}(K;\theta_1)$, $\boldsymbol{x}(K;\theta_2)$ and $\boldsymbol{m}(K;\theta_1)$ and let B denote that of the triangle with vertices $\boldsymbol{y}(K;\theta_2)$, $\boldsymbol{x}(K;\theta_1)$ and $\boldsymbol{m}(K;\theta_1)$. Then clearly

$$\int_{\theta_1}^{\theta_2} L^2(Q;\theta)d\theta = 8A.$$

We use this observation and its obvious analogues to see that

$$\begin{aligned}\int_0^{2\pi} L^2(K;\theta)d\theta &= \int_0^{2\pi} L^2(Q;\theta)d\theta - 16A + \int_0^{2\pi} L^2(R;\theta)d\theta - 16B \\ &\geqq \int_0^{2\pi} L^2(Q;\theta)d\theta + 8A(R) - 16A - 16B \\ &= \int_0^{2\pi} L^2(Q;\theta)d\theta + 8A(K) - 8A(Q)\end{aligned}$$

from which (14) follows immediately.

3. Proofs of the theorems

As we have noted, in order to establish the required inequalities it suffices to show that if $z \notin M(\mathcal{F})$ then $w(z;M(\mathcal{F})) \leqq 0$ in the cases $\mathcal{F} = \mathcal{G}$ and $\mathcal{H}$.

Lemma 6. *Let P be a convex polygon for which $M(P)$ is a simple closed curve. Then*

$$w(z, M(P)) \leqq 0$$

for all $z \notin M(P)$.

Proof. Let S and T be contiguous segments of $M(P)$ such that S and T are not collinear and let $\boldsymbol{a} = S \cap T$. We choose θ_1, θ_2 such that $\theta_1 \leqq \theta_2$ and $\boldsymbol{m}(P;\theta) = \boldsymbol{a}$ for all $\theta \in [\theta_1, \theta_2]$. We also assume θ_1, θ_2 are chosen in such a way that $\theta_2 - \theta_1$ is maximal. Then put

$$\boldsymbol{t}_1 = \lim_{n\to\infty}\left(\boldsymbol{m}(P;\theta_1) - \boldsymbol{m}\left(P;\theta_1 - \frac{1}{n}\right)\right)\left(\left\|\boldsymbol{m}(P;\theta_1) - \boldsymbol{m}\left(P;\theta_1 - \frac{1}{n}\right)\right\|\right)^{-1}$$

and

$$\boldsymbol{t}_2 = \lim_{n\to\infty}\left(\boldsymbol{m}\left(P;\theta_2 + \frac{1}{n}\right) - \boldsymbol{m}(P;\theta_2)\right)\left(\left\|\boldsymbol{m}\left(P;\theta_2 + \frac{1}{n}\right) - \boldsymbol{m}(P;\theta_2)\right\|\right)^{-1}.$$

So $\boldsymbol{t}_1$, $\boldsymbol{t}_2$ are unit vectors lying along S and T respectively. Then there must be one-sided unit tangent vectors $\boldsymbol{q}_1$, $\boldsymbol{w}_1$, $\boldsymbol{q}_2$, $\boldsymbol{w}_2$ at $\boldsymbol{x}(P;\theta_1)$, $\boldsymbol{y}(P;\theta_1)$, $\boldsymbol{x}(P;\theta_2)$, $\boldsymbol{y}(P;\theta_2)$ respectively such that

$$\boldsymbol{t}_i = (\boldsymbol{q}_i + \boldsymbol{w}_i)(\|\boldsymbol{q}_i + \boldsymbol{w}_i\|)^{-1}$$

for $i = 1,2$. For any unit vectors $\boldsymbol{u}$, $\boldsymbol{v}$ we denote by $arc(\boldsymbol{u},\boldsymbol{v})$ the open arc of the unit circle joining $\boldsymbol{u}$ and $\boldsymbol{v}$ which goes from $\boldsymbol{u}$ to $\boldsymbol{v}$ when traversed anticlockwise. We say that $M(P)$ makes an *anticlockwise turn* at $\boldsymbol{a}$ if

$$\boldsymbol{t}_2 \in arc(\boldsymbol{t}_1, -\boldsymbol{t}_1).$$

We assume initially that $\boldsymbol{q}_1 \in arc(\boldsymbol{w}_1, -\boldsymbol{w}_1)$. Then since $w_2 \in arc(\boldsymbol{w}_1, q_1)$ except when $\theta_1 = \theta_2$ and $w_1 = w_2$, we see that a necessary and sufficient condition for an anticlockwise turn at $\boldsymbol{a}$ is $\boldsymbol{q}_2 \in arc(\boldsymbol{w}_2, -\boldsymbol{w}_2)$. Now let $\boldsymbol{a} = \boldsymbol{a}_1, \boldsymbol{a}_2, \cdots, \boldsymbol{a}_n$ be a sequence of consecutive vertices of $M(P)$ at each of which $M(P)$ makes an anticlockwise turn. Then if $\boldsymbol{t}_i$ and $\boldsymbol{t}_{i+1}$ are unit tangent vectors along the segments of $M(P)$ entering and leaving $\boldsymbol{a}_i$ we can find unit tangent vectors $\boldsymbol{q}_i$, $\boldsymbol{w}_i$ such that

$$\boldsymbol{t}_i = (\boldsymbol{q}_i + \boldsymbol{w}_i)(\|\boldsymbol{q}_i + \boldsymbol{w}_i\|)^{-1}$$

for $i = 1, \cdots, n+1$. Because of the ordering of the $\boldsymbol{a}_i$ we have

$$\boldsymbol{w}_{i+1} \in arc(\boldsymbol{w}_i, \boldsymbol{w}_{i+2}) \subset arc(\boldsymbol{w}_1, \boldsymbol{q}_1)$$

and

$$\boldsymbol{q}_{i+1} \in arc(\boldsymbol{q}_i, \boldsymbol{q}_{i+2}) \subset arc(\boldsymbol{q}_1, \boldsymbol{w}_1)$$

for $i = 1, \cdots, n-1$. Also since each of the turns at the $\boldsymbol{a}_i$ is anticlockwise we have

$$\boldsymbol{q}_i \in arc(\boldsymbol{w}_i, -\boldsymbol{w}_i) \subset arc(\boldsymbol{w}_i, -\boldsymbol{q}_1)$$

and thus

$$\boldsymbol{t}_i \in arc(\boldsymbol{t}_1, \boldsymbol{q}_1^{\perp})$$

for each $i = 1, \cdots, n+1$ where $\boldsymbol{q}_1^{\perp}$ is orthogonal to $\boldsymbol{q}_1$ with

$$\boldsymbol{q}_1^{\perp} \in arc(\boldsymbol{q}_1, -\boldsymbol{q}_1).$$

Hence

$$\boldsymbol{t}_i \in arc(\boldsymbol{t}_1, -\boldsymbol{t}_1)$$

and so, in particular, $\boldsymbol{a}_n \neq \boldsymbol{a}_1$. Now assume further that $\boldsymbol{a}_0, \boldsymbol{a}_1, \cdots, \boldsymbol{a}_{n+1}$ are consecutive vertices of $M(P)$ with clockwise turns at $\boldsymbol{a}_0$ and $\boldsymbol{a}_{n+1}$ and anticlockwise turns at all other $\boldsymbol{a}_i$. So we now have, in addition, vectors $\boldsymbol{q}_0$, $\boldsymbol{w}_0$, $\boldsymbol{q}_{n+2}$, $\boldsymbol{w}_{n+2}$ with

$$t_i = (q_i + w_i)(\|q_i + w_i\|)^{-1}$$

for $i = 0$ and $n + 2$. Because of the clockwise turns at a_0 and a_{n+1} we have

$$w_i \in arc\,(q_i, -q_i)$$

for $i = 0$ and $n + 2$. But

$$w_{n+2} \in arc\,(w_1, q_1)$$

and

$$q_{n+2} \in arc\,(-w_{n+2}, w_1)$$

and so

$$t_1 \in arc\,(t_{n+2}, -t_{n+2}).$$

Consequently the total anticlockwise variation from t_1 to t_{n+1} is less than π and t_{n+2} is on the clockwise side of t_1. Similarly, since

$$q_0 \in arc\,(w_{n+1}, q_{n+1})$$

and

$$w_0 \in arc\,(q_0, -q_{n+1})$$

we have

$$t_0 \in arc\,(t_{n+1}, -t_{n+1}).$$

Thus we also see that t_{n+1} is on the clockwise side of t_0.

If we changed our initial assumption to the alternative condition that

$$w_1 \in arc\,(q_1, -q_1)$$

we would, by means of analogous arguments, come to the same conclusions concerning the total anticlockwise variation and the relative positions of t_1 and t_{n+2} and of t_0 and t_{n+1}.

These observations show that there is a sequence $S_0, S_1, \cdots, S_m$ of segments of $M(P)$ such that S_{i+1} follows S_i along $M(P)$ with the following properties:

(i) $S_m = S_0$;

(ii) if s_i is the unit tangent vector to $M(P)$ along S_i then

$$s_{i+1} \in arc\,(-s_i, s_i)$$

and $s_0 = s_m = t_0$;

(iii) the S_i are precisely the segments of $M(P)$ immediately preceding vertices at which $M(P)$ makes a clockwise turn;

(iv) any segment T of $M(P)$ lies between two consecutive members of $\{S_0, \cdots, S_m\}$, say T ($\neq S_i$) follows S_i and is followed by S_{i+1}, if t is the unit tangent vector to $M(P)$ along T then

$$t \in \overline{arc(-s_i, s_{i+1})}$$

where the horizontal line denotes closure.

Let u_i and u_{i+1} be unit tangent vectors to $M(P)$ along two consecutive line segments U_i and U_{i+1} of $M(P)$ and put $\{b_i\} = U_i \cap U_{i+1}$. If $M(P)$ makes an anticlockwise turn at b_i and $t \in arc(u_i, u_{i+1})$ we say that t *supports* $M(P)$ at b_i. Similarly if $M(P)$ makes a clockwise turn at b_i and $t \in arc(u_{i+1}, u_i)$ we say t *supports* $M(P)$ at b_i. We shall also say that u_i *supports* $M(P)$ at all points of U_i. Thus using properties (ii) and (iv) above we see that if s_0 supports $M(P)$ at a point p of $M(P)$ following a_0 (as well as at a_0, of course) then there is an i such that

$$s_0 \in arc(-s_i, s_i).$$

So if q follows p on $M(P)$ but precedes a_0 and t supports $M(P)$ at q we see that

$$q \in arc(-s_0, s_i)$$

and in particular $q \neq -s_0$.

Next we let T_0 and T_1 be the segments of $M(P)$ containing a_0 such that T_i is parallel to t_i for $i = 0$ and 1 and denote by L_0 and L_1 the lines containing T_0 and T_1 respectively. Let C denote the open cone defined by L_0 and L_1 and which does not contain T_0 or T_1 on its frontier. We shall assume there is a point $b_0 \neq a_0$ such that

$$b_0 \in M(P) \cap L_0 \cap \bar{C}$$

and seek a contradiction. In this case as we move along $M(P)$ from a_0 to b_0 we see that since both these points lie on L_0 there must be a point p between a_0 and b_0 such that s_0 supports $M(P)$ at p. Similarly there must be a point $q \in M(P)$ between b_0 and a_0 such that $-s_0$ supports $M(P)$ at q. But we have just seen that this is impossible and so we have the required contradiction. Since there is no such point b_0 we can use the simplicity of $M(P)$ to choose a point x in the interior of the convex hull of T_0 and T_1 such that if R is the ray issuing from x and containing a_0 then

$$R \cap C \cap M(P) = \emptyset.$$

Consequently we can assume that x is chosen so that

$$R \cap M(P) = \{a_0\}$$

and hence $w(x; M(P)) = -1$. But $M(P)$ is simple and so has only one bounded residual domain. Consequently $w(z; M(P)) \leqq 0$ for all $z \notin M(P)$ as required.

LEMMA 7. *Let P be a convex polygon for which $N(P)$ is a simple closed curve. Then $w(z, N(P)) \leqq 0$ for all $z \notin N(P)$.*

PROOF. The proof is very similar to that of Lemma 6 and so we only sketch the main ideas. We recall that $N(P)$ consists of a finite number of hyperbolic arcs and the area bisectors are tangents to their corresponding arcs. The vertices of $N(P)$ are the intersections of consecutive hyperbolic arcs. We say that $N(P)$ makes an anticlockwise turn at a vertex v if both the arcs at v lie on the same side of a tangent to one of them at v. Again the proof hinges on the fact that the total consecutive anticlockwise variation of a sequence of these arcs is less than π. If this were false, then because of the tangential properties of the area bisectors we would obtain two distinct parallel area bisectors, which is clearly impossible. The proof can now be completed by finding a point with negative winding number exactly as before.

It follows from Lemmas 6 and 7 that if $\mathscr{F} = \mathscr{G}$ or $\mathscr{H}$ and P is a polygon for which $M(\mathscr{F})$ is simple then

$$w(z, M(\mathscr{F})) \leqq 0 \tag{16}$$

for all $z \notin M(\mathscr{F})$. Combining this with equations (12) and (13) we see that (16) holds for all polygons P. Consequently Lemma 1 shows that

$$8A(K) \leqq \int_0^{2\pi} \sigma^2(\theta) d\theta \tag{17}$$

in the case K is a polygon and $\mathscr{F} = \mathscr{G}$ or $\mathscr{H}$. But any planar convex set can be approximated by means of polygons and so Lemma 3 ensures that (17) holds for arbitrary convex sets K. Thus we have obtained the inequalities of our theorems and it remains to characterize the case of equality.

So we assume for $\mathscr{F} = \mathscr{G}$ or $\mathscr{H}$ that for some convex set K

$$8A(K) = \int_0^{2\pi} \sigma^2(\theta) d\theta \tag{18}$$

and aim to show that K is centrally symmetric. Clearly the converse is true. First we consider the case where $M(\mathscr{F})$ is a simple closed curve. Then providing it is not a single point we can find a closed disc D contained in the bounded residual domain of $M(\mathscr{F})$. Let $(P_n)_{n=1}^{\infty}$ be a sequence of polygons with $P_n \to K$ as $n \to \infty$. Let $\mathscr{G}_n$ ($\mathscr{H}_n$) denote the family of perimeter (area) bisectors of P_n and let $\mathscr{F}_n = \mathscr{G}_n$ or $\mathscr{H}_n$ according as $\mathscr{F} = \mathscr{G}$ or $\mathscr{H}$ and denote by σ_n the length of a typical number of $\mathscr{F}_n$. Then we use Lemmas 1 and 4 together with (11) and (16) to deduce that for sufficiently large values of n

$$8A(P_n) - \int_0^{2\pi} \sigma_n^2(\theta)d\theta = 8\int_{z \notin M(\mathscr{F}_n)} w(z; M(\mathscr{F}_n))dz$$

$$\leqq 8\int_{z\in D} w(z; M(\mathscr{F}_n))dz$$

$$\to -8A(D)$$

as $n \to \infty$. Thus, by Lemma 3

$$8A(K) - \int_0^{2\pi} \sigma^2(\theta)d\theta \leqq -8A(D) < 0$$

which contradicts (18). Hence if K satisfies (18), $\mathscr{F} = \mathscr{G}$ or $\mathscr{H}$ and $M(\mathscr{F})$ is a simple closed curve it must be a single point. Hence all the perimeter (or area) bisectors bisect each other. So it follows from [10] that K is centrally symmetric.

Consequently we now assume that $M(\mathscr{F})$ is not simple. So we can find angles θ_1', θ_2', θ_0 with $\theta_1' < \theta_0 < \theta_2'$ and

$$\boldsymbol{m}(K; \theta_1') = \boldsymbol{m}(K; \theta_2') \neq \boldsymbol{m}(K; \theta_0).$$

Put

$$\delta = \sup\{\phi - \theta : \theta_1' \leqq \theta < \phi \leqq \theta_0,\ \boldsymbol{m}(K;\theta) = \boldsymbol{m}(K;\phi) \text{ and there is a } \lambda \in (\theta,\phi) \text{ with } \boldsymbol{m}(K;\lambda) \neq \boldsymbol{m}(K;\theta)\}.$$

If $\delta > 0$ we may choose α_1, β_1 with $\theta_1' \leqq \alpha_1 < \beta_1 \leqq \theta_0$, $\boldsymbol{m}(K;\alpha_1) = \boldsymbol{m}(K;\beta_1)$ and $\beta_1 - \alpha_1 = \delta$. Then clearly $\boldsymbol{m}(K;\theta) \neq \boldsymbol{m}(K;\alpha_1)$ if $\theta \notin [\alpha_1, \beta_1]$. We put $K_1 = K(\alpha_1, \beta_1)$ and

$$\delta_1 = \sup\{\phi - \theta : \theta_1' \leqq \theta < \phi \leqq \theta_0,\ \boldsymbol{m}(K_1;\theta) = \boldsymbol{m}(K_1;\phi) \text{ and there is a } \lambda \in (\theta,\phi) \text{ with } \boldsymbol{m}(K_1;\lambda) \neq \boldsymbol{m}(K_1;\theta)\}.$$

Again if $\delta_1 > 0$ we may choose α_2, β_2 with $\theta_1' \leqq \alpha_2 < \beta_2 \leqq \theta_0$, $\boldsymbol{m}(K_1;\alpha_2) = \boldsymbol{m}(K_1;\beta_2)$ and $\beta_2 - \alpha_2 = \delta$. We note that

$$(\alpha_1, \beta_1) \cap (\alpha_2, \beta_2) = \varnothing$$

and put $K_2 = K_1(\alpha_1, \beta_1)$. Continuing in this fashion we generate an at most countable collection of sets $K_1, K_2, \cdots$ and either $\delta_n = 0$ for some n or $\delta_n \to 0$ as $n \to \infty$. Because of the Blaschke Selection Theorem we may assume $K_n \to K^*$ as $n \to \infty$. Then clearly $M(K^*) \subset M(K_n)$ for all n or $N(K^*) \subset N(K_n)$ for all n according as $\mathscr{F} = \mathscr{G}$ or $\mathscr{H}$. We shall let $\mathscr{G}^*$ ($\mathscr{H}^*$) denote the family of perimeter (area) bisectors of K^* and put $\mathscr{F}^* = \mathscr{G}^*$ or $\mathscr{H}^*$ according as $\mathscr{F} = \mathscr{G}$ or $\mathscr{H}$. Now assume there are θ, λ, ϕ with $\theta_1' \leqq \theta < \lambda < \phi \leqq \theta_0$ and such that

$$\boldsymbol{m}(K^*;\theta)=\boldsymbol{m}(K^*;\phi)\neq\boldsymbol{m}(K^*;\lambda).$$

We shall seek a contradiction and thus conclude that the arc of $M(\mathcal{F}^*)$ between $\boldsymbol{m}(K^*;\theta_1')$ and $\boldsymbol{m}(K^*;\theta_0)$ is a simple path. If

$$\theta\notin\bigcup_{i=1}^{\infty}(\alpha_i,\beta_i)$$

then $\boldsymbol{m}(K_i;\theta)=\boldsymbol{m}(K;\theta)$ for all i. If $\theta\in(\alpha_i,\beta_i)$ then $\boldsymbol{m}(K_j;\theta)=\boldsymbol{m}(K_i;\theta)$ for all $j\geqq i$. Applying this argument to λ and ϕ we see that there is an I such that for $\mu=\theta$, λ and ϕ we have

$$\boldsymbol{m}(K^*;\mu)=\boldsymbol{m}(K_i;\mu)$$

for all $i\geqq I$. But then $\delta_i\geqq\phi-\theta$ for all $i\geqq I$ contradicting the fact that $\delta_i\to 0$ as $i\to\infty$. Consequently if $\boldsymbol{m}(K^*;\theta)=\boldsymbol{m}(K^*;\phi)$ with $\theta_1'\leqq\theta<\phi\leqq\theta_0$ then $\boldsymbol{m}(K^*;\lambda)=\boldsymbol{m}(K^*;\theta)$ for all $\lambda\in(\theta,\phi)$. But by Lemma 2, $M(\mathcal{F}^*)$ is rectifiable and so if we parametrize it by means of arc length we see that the arc of $M(\mathcal{F}^*)$ between $\boldsymbol{m}(K^*;\theta_1')$ and $\boldsymbol{m}(K^*;\theta_0)$ is a simple path. We also conclude from Lemmas 3 and 5 together with (18) that

$$8A(K^*)=\int_0^{2\pi}\sigma_*^2(\theta)d\theta$$

where σ_* denotes the length of a typical number of $\mathcal{F}^*$.

Applying the same arguments to the interval $[\theta_0,\theta_2']$ we may assume that we have a convex set K such that

$$8A(K)=\int_0^{2\pi}\sigma^2(\theta)d\theta$$

and that there are angles θ_1', θ_2', θ_0 with

$$\boldsymbol{m}(K;\theta_1')=\boldsymbol{m}(K;\theta_2')\neq\boldsymbol{m}(\boldsymbol{K};\theta_0)$$

such that the arc of $M(\mathcal{F})$ between $\boldsymbol{m}(K;\theta_1')$ and $\boldsymbol{m}(K;\theta_0)$ and the arc between $\boldsymbol{m}(K;\theta_0)$ and $\boldsymbol{m}(K;\theta_2')$ are both simple paths. We shall denote these simple paths by P_1 and P_2 respectively.

Our next aim is to show that $P_1\neq P_2$ where we just consider the P_i as point sets. To do this we assume $P_1=P_2$ and seek a contradiction.

If $\mathcal{F}=\mathcal{G}$ and $P_1=P_2$ we see that for each $\theta_1\in(\theta_1',\theta_0)$ there is a $\theta_2\in(\theta_0,\theta_2')$ with

$$\boldsymbol{m}(K;\theta_1)=\boldsymbol{m}(K;\theta_2).$$

If t is a one-sided tangent to $M(K)$ at $m(K;\theta_1)$ then $-t$ is a one-sided tangent to $M(K)$ at $m(K;\theta_2)$. Now let t_1, t_2, t_3, t_4 be the corresponding one-sided tangents to K at $x(K;\theta_1)$, $y(K;\theta_1)$, $x(K;\theta_2)$, $y(K;\theta_2)$ respectively. If t_1 is not parallel to t_2 and t_3 is not parallel to t_4 we have

$$t = (t_1 + t_2)(\|t_1 + t_2\|)^{-1} \quad \text{and} \quad -t = (t_3 + t_4)(\|t_3 + t_4\|)^{-1}. \tag{19}$$

But the anticlockwise ordering of these unit vectors on the unit circle is

$$t_1, t_3, t_2, t_4, t_1$$

and this clearly contradicts (19). Hence either t_1 is parallel to t_2 or t_3 is parallel to t_4.

If t_1 is not parallel to t_2 we can find angles α_1, α_2 close to θ_1 with $m(K;\alpha_1) \neq m(K;\alpha_2)$ and such that if $\theta \in (\alpha_1, \alpha_2)$ then K does not have parallel support lines at $x(K;\theta)$ and $y(K;\theta)$. We choose β_1, β_2 close to θ_2 such that

$$m(K;\alpha_i) = m(K;\beta_i)$$

for $i = 1,2$ and so that the sub-path of P_1 from $m(K;\alpha_1)$ to $m(K;\alpha_2)$ is the same (except for orientation) as the subpath of P_2 from $m(K;\beta_2)$ to $m(K;\beta_1)$. Then our above observations show that K has parallel support lines at the points $x(K;\phi)$ and $y(K;\phi)$ for all $\phi \in (\beta_2, \beta_1)$. It then follows from the proof of theorem 2.2 of [5] that $m(K;\beta_1) = m(K;\beta_2)$ which contradicts our choice of α_1 and α_2. Consequently we must have both t_1 parallel to t_2 and t_3 parallel to t_4. But this must be true for any choice of θ_1 in (θ_1', θ_0) and so the above argument now shows that $m(K;\theta_1') = m(K;\theta_0)$. This contradiction thus proves that $P_1 \neq P_2$ in the case $\mathcal{F} = \mathcal{G}$.

Now assume $\mathcal{F} = \mathcal{H}$ and $P_1 = P_2$. We recall that if the tangents to K at $x(K;\theta)$ and $y(K;\theta)$ are not parallel then the tangent to $N(K)$ at $m(K;\theta)$ is parallel to $x(K;\theta) - y(K;\theta)$. So we deduce that for each $\theta \in (\theta_1', \theta_0)$, K must have parallel support lines at $x(K;\theta)$ and $y(K;\theta)$. This time we use the proof of theorem 2.4 of [5] to obtain $m(K;\theta_1') = m(K;\theta_0)$. So again we can use this contradiction to deduce that $P_1 \neq P_2$.

If $\mathcal{F} = \mathcal{G}$ or $\mathcal{H}$ we can assume without loss of generality that $P_1 \setminus P_2 \neq \emptyset$. So we can choose θ_1, θ_2 with $\theta_1' \leqq \theta_1 < \theta_2 \leqq \theta_0$ such that $m(K;\theta_1) \neq m(K;\theta_2)$ and $m(K;\theta) \in P_1 \setminus P_2$ for all $\theta \in (\theta_1, \theta_2)$. We can of course do this in such a way that there are $\theta_3, \theta_4 \in [\theta_0, \theta_2']$ with $m(K;\theta_1) = m(K;\theta_3)$ and $m(K;\theta_2) = m(K;\theta_4)$. There are now two possibilities to consider: (i) $\theta_3 > \theta_4$ and (ii) $\theta_3 < \theta_4$.

In case (i) we put $C = K(\theta_2, \theta_4)$ and $D = C(\theta_3, \theta_1 + \pi)$. Let $\mathcal{G}(D)$ (resp. $\mathcal{H}(D)$) denote the family of perimeter (resp. area) bisecting chords of D and put

$\mathcal{F}(D)=\mathcal{G}(D)$ or $\mathcal{H}(D)$ according as $\mathcal{F}=\mathcal{G}$ or $\mathcal{H}$. Then $M(\mathcal{F}(D))$ is a simple closed curve contained in $M(\mathcal{F})$ and preserving the orientation of $M(\mathcal{F})$. In fact

$$M(\mathcal{F}(D))=\{\boldsymbol{m}(K;\theta):\theta\in[\theta_1,\theta_2]\cup[\theta_4,\theta_3]\}.$$

Also if σ_D denotes the length of a typical member of $\mathcal{F}(D)$ we see that repeated use of Lemma 5 in conjunction with (18) gives

$$8A(D)=\int_0^{2\pi}\sigma_D^2(\theta)d\theta.$$

But $M(\mathcal{F}(D))$ is simple and so our earlier observations show that D is centrally symmetric and thus that $M(\mathcal{F}(D))$ is a single point. This contradicts the fact that

$$\boldsymbol{m}(D;\theta_1)=\boldsymbol{m}(K;\theta_1)\neq\boldsymbol{m}(K;\theta_2)=\boldsymbol{m}(D;\theta_2)$$

and so case (i) cannot occur.

In case (ii) we put

$$\theta_5=\max\{\theta\in[\theta_0,\theta_3]:\boldsymbol{m}(K;\theta)=\boldsymbol{m}(K;\phi)\text{ for some }\phi\in[\theta_2,\theta_0]\}$$

and choose $\theta_6\in[\theta_2,\theta_0]$ such that $\boldsymbol{m}(K;\theta_6)=\boldsymbol{m}(K;\theta_5)$. Then clearly

$$\theta_2\leqq\theta_6\leqq\theta_0\leqq\theta_5\leqq\theta_3.$$

Now as in case (i) we can construct a convex set D for which $M(\mathcal{F}(D))$ is the simple closed curve

$$\{\boldsymbol{m}(K;\theta):\theta\in[\theta_1,\theta_6]\cup[\theta_5,\theta_3]\}.$$

Again Lemma 5 and (18) show that

$$8A(D)=\int_0^{2\pi}\sigma_D^2(\theta)d\theta$$

and so the fact that $M(\mathcal{F}(D))$ is a singleton shows that case (ii) cannot occur.

Consequently we have shown that for (18) to hold our set K must be centrally symmetric and so our theorems are established.

REMARK. The following observation was recently brought to my attention by E. Lutwak to whom I am very grateful. If we use the notation

$$M_p(L,K)=\left\{\frac{1}{2\pi}\int_0^{2\pi}L^p(K;\theta)d\theta\right\}^{1/p}$$

for $p\in\mathbf{R}\setminus\{0\}$ then our first theorem can be restated in the form

$$M_2(L,K)\geqq 2\sqrt{(A(K)/\pi)}$$

with equality if and only if K is centrally symmetric. For $p = 0$ or $\pm\infty$ we put

$$M_p(L, K) = \lim_{s \to p} M_s(L, K).$$

It is well-known (see G. H. Hardy, J. E. Littlewood and G. Polya, *Inequalities*, Cambridge University Press, 1934) that $M_p(L, K)$ is continuous and monotone increasing in p and that

$$M_\infty(L, K) = \max\{L(K; \theta) : \theta \in [0, 2\pi]\}$$

while

$$M_{-\infty}(L, K) = \min\{L(K; \theta) : \theta \in [0, 2\pi]\}.$$

We immediately deduce that

$$M_p(L, K) \geqq 2\sqrt{(A(K)/\pi)}$$

for all $p \geqq 2$. In case $p > 2$ it also follows that equality occurs here if and only if K is a disc. Finally we note that if $p < 2$ and K is any centrally symmetric set except a disc we have

$$M_p(L, K) < M_2(L, K) = 2\sqrt{(A(K)/\pi)}.$$

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